# Linear Social Network Models

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proposed by Becker and Murphy (2001) when the consideration of social factors first began to play a major role in economic analyses.

Within the broad area of social economics, the study of social networks has arguably become the most prominent area of research. From the perspective of economic theory, social network analysis is now a well established area of specialization; see Jackson (2008) and Goyal (2009) as well as the relevant chapters in Benhabib, Bisin, and Jackson (2010) for overviews of the existing theory. Further, it is common for empirical papers to invoke social networks as the underlying structure by which social influences are transmitted; one standard example is information transmission.

While the theoretical and empirical literatures on social networks have expanded greatly over the last decade, there has been little contact between them. There has been remarkably little work on formal econometric issues pertaining to identification. This is especially surprising since there is now a rich literature on identification problems in social interactions models-a subset of social network models which assumes that individuals belong to predefined groups wherein all group members influence each other with equal intensity.[2]

The identification literature on social interactions has, since Manski (1993), demonstrated that important limits exist to identification. In particular, the so-called linear-in-means model, the workhorse of empirical research on social interactions,

Fortin provide a condition on the matrix defining social interactions which is sufficient for identification of the parameters in the model they consider. They also demonstrate how

endogenous and contextual-effects transmission mechanisms are better grounded in theories of social influence and can have different statistical properties that provide alternative paths to identification.

Second, the impact of endogenous network construction is rarely addressed in systematic fashion. Structural models of network formation are typically not linked to behavior within networks. Instead network endogeneity is addressed using instrumental variables are employed whose validity is often unclear when one considers network formation and behavior in networks as two stages of a game.

Third, there is, to the best of our knowledge, there exists no systematic discussion of the informational content of aggregated data for network effects. The few studies examining aggregate data, most notably Glaeser, Sacerdote and Scheinkman (1996) and Graham (2008), focus on the use of aggregate data to estimate particular parameters or provide evidence of some type of social effect rather than assess overall information content.

Finally, while Blume, Brock, Durlauf and Ioannides provide examples of identification under partial network observability, this question has received very little attention in econometrics. A partial exception is Conley and Topa (2003) who explore mismeasurement of groups in social interactions models.

This paper provides a systematic analysis of identification in linear social networks models. This is both a theoretical and an econometric exercise in that it links identification analysis to a rigorously delineated model of interdependent decisions. The paper proceeds as follows. Section 2 describes a social network game from which the linear model emerges as a unique equilibrium. This section introduces different mechanisms for the spread of endogenous and contextual effects through the network. Section 3 provides the conceptual framework which we use to study identification. Section 4 explores identification results when network structure is both exogenous and known *a priori*. It considers how differences in the spread of contextual and peer effects can aid identification. It also addresses identification from aggregate data. Section 5 considers identification when the network is exogenously given but is not observed by the econometrician. Section 6 examines endogenous network formation. The theoretical model of Section 2 is extended to a simple network formation game. Two

different models of preferences for networks are discussed, and identification conditions are developed. The possibility of extending control function techniques to account for network endogeneity is discussed. Section 7 concludes. A technical appendix follows which contains all proofs.

## 2. A social networks game

In this section we provide an explicit derivation of a linear social networks model for individual behavior as the *Bayes-Nash equilibrium* for a social networks game and demonstrate that these linear behavioral rules are the unique descriptions for individual behavior. An analogous result is developed for social interactions models in Blume, Brock, Durlauf, and Ioannides (2010). As is the case for the earlier derivation, the unsurprising key to justifying linear social networks models as econometric specifications for individual behavior is to assume that individual agents possess quadratic payoff functions. As such our model is a species of quadratic interaction games that have become popular in different contexts.

### i. the quadratic social networks model

The social networks game we describe is a game of incomplete information. In this game, each individual is described by a bundle of characteristics, some observed by everyone, including the econometrician, some observed by the individuals in the population but not by the econometrician, and some private to the individual. Individuals have preferences over their actions, which are the sum of a private component and a social component.[5]

[5]Game-theoretic models are usually interpreted to have individual preferences over outcomes,

This payoff function takes as special cases many of the payoff functions that have at least implicitly appeared in the literature. As far as we know our analysis is the first fully rigorous demonstration of existence and uniqueness of an equilibrium for a general quadratic social networks model.

The first two terms constitute the component of the payoff function that is independent of the choices of others; we call this the private component of the payoff function. The marginal value of the individual's choice depends upon his characteristics and a weighted average of the characteristics of others, computed with the weights $c_{ij}$. The matrix of these weights is $C$. This matrix is assumed to be nonnegative and each row sums to 1 this generalizes the common assumption in the social interactions literature that the unweighted average of individual characteristics affects each member of a group. The final component captures a purely social component to payoffs in that the component depends on the choices of others. It is quadratic and decreasing in the squared distance between the individual's choice and a weighted average of the choices of others, computed with weights $a_{ij}$. The matrix of these weights is $A$. The parameter determines the weight placed on the public component relative to the private component. Accordingly, 0. Note that if the values of the $c_{ij}$'s and the $a_{ij}$'s are known *a priori*, there are $2P$ 1 utility parameters which determine choice.

The private component includes a conventional externality, that network average characteristics are a kind of group capital or public good. This is expressed in the term $\sum_{p=1}^{P} \sum_{j} c_{ij} x_j^p$. The literature refers to this term as the *contextual effect*. Here we have extended the idea from groups, and group averages, to networks. Contextual effects for the entire network are thus summarized by the sociomatrix $C$.[7] The literature on social interactions presents two kinds of contextual variables: group averages of individual level variables, and distinct group variables. In a study of peer effects on educational outcome, for instance, classroom-specific average family income and teacher-specific variables are examples of the first and second type, respectively. It has been known

[7]Sociomatrix is a term from sociology. In the mathematics literature these objects are called weighted adjacency matrices.

since Brock and Durlauf (2001a,b) that the relationships between these variables determine whether identification holds. In particular, they show in a related model that the presence of one individual variable whose group average is not a contextual variable is sufficient for identification. Since they raise no new issues here, we omit group variables which are not averages of individual variables.

The social component captures the idea that deviating from the average behavior of one's peers is costly. This is the source of so-called peer, or endogenous, effects. As in the case of the contextual effects, the weights $a_{ij}$

observations $x_i$ are stacked variable by variable, with the individual-specific vectors so created stacked as well.

Our various assumptions may be summed up by the following axioms.

**Axiom 1.** 0, $A$ and $C$ are non-negative, their row sums are 1, and for all $i$, $a_{ii}$ 0.

**Axiom 2.** Second moments of the marginal distribution exist.

**Axiom 3.** Second moments of the marginal distribution exist.

**Axiom 4.** For all $i$, $E$ $_i|x,$ , $_i$ $_i$ is independent of $x$ and $_i$.

Axiom 1 restricts the payoff function.[8] The second and third axioms guarantee that the choice problems required of individuals by the game are well-posed, that the necessary expectations exist. The joint distribution on $x$, and is the common prior belief on the space of types. If Axiom 4 were false, the equilibrium strategies need not be linear in $x$.

A strategy for individual $i$ is a function that assigns an action to each of his possible types, a function $f_i : \mathbf{R}^{|V|\ P\ 1\ 1}$ **R**. A Bayes-Nash equilibrium BNE of the game is a strategy profile $_i$

$$f_i^*(x, \ , \ _i \ ^{|V|}_{i\ 1}$$

$$\frac{1}{1} I \ \frac{}{1} A^{\ 1} \ \ \ \ \ \ \ \ ^{P}_{p\ 1} \ ^{p}I \ \ ^{p}C \ x^{p} \ \ \frac{1}{1} \ ^{dev}_{i} \qquad (2)$$

where $^{dev}_{i}$ $_i$ $_i$ .

Notice that strategy profile implies distinct roles for $_i$ and $^{dev}_{i}$ since only the former is common knowledge.

The equilibrium strategies map types into actions; that is, strategies describe a map $f$ : $x$, , . This is what the empirical literature calls a reduced form. This is, in fact, the structural model for a game theorist insofar as what one means by a structural model is a model delivered from theoretical considerations. In the social networks and social interactions literature, what are called structural models are equations in which individual choices are determined by the individual choices (or expected choices) and characteristics of others as well as the characteristics of individuals. At best, these may be first-order conditions. In our model, for example, the first order conditions for utility maximization are, for all

as the structural model (which it is) or a reduced form (which it is often called, but is not), but instead as the strategy profile of the network. We will abuse this term slightly, because we will also use the term strategy profile to refer to the matrix $B \quad B_1,...,B_P$ in which each matrix $B_p$ acts on the characteristics $x^p$ and is given by the function $_p \quad , \ , \ ,C,A$ defined as follows:

$$B_p \quad _p \quad , \ , \ ,C,A \quad \frac{1}{1} \ I \ \frac{}{1} A \quad ^1 \quad ^pI \quad ^pC \ . \qquad (4)$$

Notice that Axiom 1 implies that the matrix inverse on the right hand side exists for all $0$. From equation (2) it can be seen that the matrix $B$ essentially characterizes the equilibrium strategies, and our identification exercises essentially involve determining what parameters can be recovered from them. If the span of $x^p$ has dimension less than $|V|$, then $B_p$ will be unique only up to its action on a lower-dimensional subspace. But this is an issue of identification, not one of existence or uniqueness.

### ii. social interactions models as special cases of the general linear social networks model

The social interactions literature has focused on the special case where individuals react to the average of others in a predefined group $g$. Notationally, $g$ denotes a collection of indices corresponding to population members. Social interactions models assume that each member of a group is affected by the average behavior of others in the group and is unaffected by individuals who are not members of the group. Following Blume, Brock, Durlauf, and Ioannides (2010), the microfounded quadratic social interactions model is a special case of our social networks model such that[9]

[9]Bramoullé, Djebbari, and Fortin refer to the case where $i$ is omitted from the averaging as exclusive averaging and associate this form of averaging with Moffitt (2001). They

$$c_{ij} \quad \frac{1}{n_g} \qquad \text{if } i, j \quad g,$$

$$a_{ij} \quad \frac{1}{n_g \quad 1} \qquad \text{if } i, j \quad g,\ i \quad j \tag{5}$$

$$c_{ij} \quad a_{ij} \quad 0 \qquad \text{otherwise}$$

where $n_g$ is the size of group $g$. Under these restrictions, the first order condition for an individual's choice produces the first order condition (interpreted as a structural equation in the literature)[10]

$$_{i} \quad \frac{P \quad p}{_{p\ 1}\ 1} x_i^p \quad \frac{P \quad p}{_{p\ 1}\ 1 \quad n_g \quad _j} x_j^p \quad _{j\ i} \frac{}{1 \quad n_g \quad 1} E \quad _j \Big| x \quad \frac{1}{1} \quad _i \tag{6}$$

When the population size is large, this expression becomes arbitrarily close to

$$_{i} \quad \frac{P \quad p}{_{p\ 1}\ 1} x_i^p \quad \frac{P \quad p}{_{p\ 1}\ 1} \bar{x}_g^p \quad \frac{}{1} E \quad _g \Big| x \quad \frac{1}{1} \quad _i \tag{7}$$

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contrast this with

where the barred variables are group averages. This last equation defines the linear-in-

iii. $_i$ is independent of $i$.

iv. For all $i$ and $j$, $a_{ij}$ 0 if and only if $a_{ji}$ 0.

v. For all $i$ and $j$, there is a pair $i$ $j$ with $c_{ij}$ 0, and $c_{ij}$ 0 if and only if $c_{ji}$ 0.

vi. One of and is not 0.

These properties are further assumptions on the theoretical model that address identification issues rather than existence and uniqueness of the equilibrium. Recall that we have already assumed that is independent of $x$, and that the $a_{ii}$ are 0. Condition i ensures that $B$ is unique, that the relevant space on which strategies are defined is full-dimensional. Conditions ii and iii shrink the size of the parameter space considerably, and can be justified as consequences of assumptions such as exchangeability that may be employed in any event. Conditions iv and v of the

will assume that first moments of , conditional moments of , and itself are all *a priori* knowledge. One use of *a priori* knowledge assumptions is to define particular parameter restrictions. For instance, in some of our theorems it is known *a priori* that 0 (no contextual effects). We will also be interested in generic identification. This too can be expressed in terms of *a priori* knowledge.

**Definition 2.** The parameter $p$ in model $M$ is generically identified from the joint distribution of and $x$ if and only if there is a closed and lower-dimensional set $M^{crit}$ such that if the complement of $M^{crit}$ is known *a priori*, then

In this way the parameters , , , , $A$ and $C$, and the distribution of $^{dev}$ and

$^{dev}$, the deviations of and(et)2(ma)3(s)]TJ 0 -1.04 Td [(n)-1(d)-1(C)-1(T)-0.6(i)-1(m)-1(e)-1

something from variances requires characteristics known to the public but not to the econometrician

A particular instance of this is Graham's (2005,2008) variance-contrast method. Graham (2005) provides a theoretical foundation for this econometric model, which is a game of complete information but with individual characteristics unobservable to the econometrician. Imposing the parameter restrictions 0 in the game of section 2 gives a game of complete information where characteristic is observed by individuals but not by the econometrician. (For Graham realized choices are observed by each agent.) Assuming further $A \quad C$ and that the social network is the union of cliques of different sizes and that each individual weighs everyone in his group equally, Graham establishes identification of .

Aside from this type of information structure, however, variance and higher-moment methods add nothing to the possibilities for identification. Variances augment identification if there are some variables that $m$ acts upon and which the econometrician does not observe. Otherwise the econometrician is left with var which reveals little of interest.

## 4. Identification with known sociomatrices

In this section we consider identification when both $A$ and $C$ are both known *a priori*. The goal here is to study the traditional reflection problem, that is, identification in the presence of contextual effects. If $A \quad C$, the analysis with individual-level data is a straightforward extension of Blume, Brock, Durlauf, and Ioannides (2010). The discussion of aggregation is new.

### i. identification with individual-level data

Recall that the sums of the means of the two unobserved variables are identified in this and all subsequent models. This is a trivial point in light of the fact that the sum of

the means of the unobservables is nothing more than the constant term in the individuals' strategies.

The following lemma is useful for checking identification when the peer- and contextual-effects network stand in particular relationships to each other.

**Lemma 2.** If $A$ and $C$ are *a priori* knowledge, if $m$ $m$ and if there is a pair $i$ $j$ such that $c_{ij}$ 0 and $b_{ij}$ 0, then , , , ,

The next theorem gives an example of how the lemma can be employed, giving one set of assumptions on the primitives which fulfill the conditions of the lemma.

**Theorem 2.** Suppose that the following facts are known *a priori*:

i. $A$ and $C$;

ii. the peer-effects network is connected;

iii. there is no individual $j$ such that $c_{kj}$ 0 for all $k$

iv. there is some pair $i, j$

individual variable whose group average is not a contextual variable. This requires some structure on the contextual effects network. The network is already assumed to be bidirectional, that $i$ has a contextual effect on $j$ if and only if $j$ has a contextual effect on $i$. We will assume in addition that the contextual effects network is transitive: If $i$ is affected by $j$, and $j$ is affected by $k$, then $i$ is directly affected by $k$. Formally, if $c_{ij}$ 0 and $c_{jk}$ 0, then $c_{ik}$ 0.

**Theorem 3.** Suppose that the following facts are known *a priori*:

i. $A$ and $C$;

ii. the contextual effects network is transitive;

iii. there are components $V_1^C$ and $V_2^C$ of the contextual-effects network and component $V^A$ of the peer-effects network such that each $V_i^C$ $V^A$ .

Then , and are identified from the joint distribution of and $x$.

Bramoullé, Djebbari, and Fortin (2009) provide a powerful identification requirement for the traditional linear-in-means model that provides a connection between identification and network structure. The next result extends this to our two-sociomatrix model.

**Lemma 3.** Suppose that $A$ and $C$ are known *a priori*.

i. Suppose it is known *a priori* that $A$ $C$ and $AC$ $A$,$C$. The matrices $I$, $A$, $C$ and $AC$ are linearly independent if and only , and are identified from the joint distribution of and $x$.

ii. Suppose it is known *a priori* that $A \quad C$ and $AC \quad C$ and that the matrices $I$, $A$, and $AC$ are linearly independent. Then , and are identified from the joint distribution of and $x$.

iii. Suppose it is known *a priori* that $A \quad C$ and that 0 Then *a priori* knowledge that $I$, $A$, and $A^2$ are independent is necessary and sufficient for , and to be identified from the joint distribution of and $x$.

The condition that 0 ensures that peer and contextual effects do not cancel

### ii. aggregation

Classroom-level and village-level data often come aggregated. For example, an education data set may contain observation on mean outcome and mean characteristics of many classrooms. What can be learned in this case? The answer is, not very much. In general, with only mean characteristics and mean outcome data, identification will be complicated by the fact that there is no direct relationship between mean characteristics and mean outcome. A special case where there is such a relationship arises when the sociomatrices are bistochastic, that is column sums as well as row sums all equal one. Although this is a very restrictive condition, it includes the important case of equal-weighted averages of all other individuals. This is the case which Graham (2008) uses

all ot to show how of all sp h 40 o ay B ase w ? 2 ( ) ] J B [ 2 ( ) ( ) 2 ( i ) e o ) 8 y o c e / 2 1 1 m a n s w e r i e 0 4 6

are non-zero will be bistochastic. Inverses and products of bistochastic matrices are also bistochastic and in particular $1 \quad {}^{1} I \quad / 1 \quad A^{n} \quad {}^{1}$ is bistochastic. As before, let bars represent group averages. Let bars represent group averages. The average outcome in group $n$, is

$$e^{n} \quad e^{n} \quad \frac{1}{1} \; I \quad \frac{}{1} A^{n} \quad {}^{1} \quad x^{n} \quad C^{n} x^{n} \quad \frac{1}{1} e^{n} \quad {}^{dev}$$

$$\overline{x}^{n} \quad \frac{1}{1} \; {}^{-dev}.$$

The obvious result is that only the sums $_{0} \quad _{0}$ and are identified.

**Theorem 4.** If only $A$ and $C$ are known *a priori*, then only and $_{0} \quad _{0}$ are identified from the joint distribution of groups, average choices, and average characteristics. No other parameters are identified.

We saw in Section 3.ii that these parameter sums are identified with individual observations, but this result is not generally true for models when only aggregate data are observed.

**b. exchangeable individuals**

The analysis of identification from aggregate data for more general classes of social networks will require assumptions on the relationship between characteristic means and the distribution of characteristics among the population. Suppose that the distribution of characteristics has the following property:

**Property P**: E $x^{n} \left| e^{n} \right. \; x^{n} \quad z \quad z, \quad , z$ .

Property P says that the conditional mean characteristic of each group member given

In this section we consider cases wherein the contextual effects sociomatrix is *a priori* knowledge, but the peer-effects sociomatrix is unknown to the econometrician. Although it may seem surprising that one can identify the peer-effects sociomatrix conditional on knowledge of the contextual-effects sociomatrix, a moment's reflection shows why it is plausible. The dimension of the set of peer effects matrices is $|V|\,|V|$ 2 . The dimension of the set $m : C$ $C^*$ for a fixed $C^*$ is no more than $|V|\,|V|$ 1 1, but we can show it to be no less than $|V|\,|V|$ 2 . We need to recover $|V|\,|V|$ 2 3 parameters from $m$ . It is certainly plausible that a necessary order condition is satisfied. We are searching for sufficiency, however, and we will proceed by direct argument rather than by trying to pin down more carefully the structure of $M$ .

**a. identification without contextual effects**

It is common in the theoretical econometrics literature to assume that the peer-effects network is known, and in the empirical literature to pretend that it is. This is rarely the case, so it is important to see how far one can go without such knowledge. The first result concerns identification when it is known a priori that 0

problem. Amazon's book pricing problem concerns networks. Are patterns of book demand due to homophily or a network effect? Price is a contextual variable, but since it is not an average of customers' characteristics, it creates no identification problems.

**Theorem 5**. If the econometrician knows *a priori* that 0, then the parameter is identified from the joint distribution of outcomes and characteristics. If 0, then and $A$ are identified.

When 0, all variation in outcomes is due to variation in the unobservable variables. $_i$ and $_i$. With further assumptJ EMC l2 such as independence, made on these terml2 we conjecture that the matrix $A$ could perhaps be discerned.

This theorem claiml that the utility parameter governing the strength of the social interaction is identified when the peer-effects matrix is not known. Moreover, the peer-effects matrix itself is, in principle, recoverable from the data. This result is surprising to us, and the rest of this section will be concerned with how far this result can be pushed.

**b. identification with contextual effects**

When contextual effects are present and the contextual-effects sociomatrix is known, it will turn out that parameter values are generically identified, and even when they are not, is identified. We will assume it to be known *a priori* that $I$ $C$ is invertible. For a given $C$, the set of , pairs for which invertibility fails is the union of a finite set of one dimensional spaces.

**Theorem 6.** If the sociomatrix $C$ and the matrix $I$ $C$ both have full rank, are known *a priori*, then and are identified from the joint distribution of outcomes and characteristics. There is a set $C$ of matrices whose complement in the set of all contextual-effects sociomatrices is closed and lower-dimensional, such that if $C$ C,

then , and are identified from the joint distribution of outcomes and characteristics.

It will be clear from the proof of this Theorem that if and cannot be distinguished, the peer-effects network cannot be identified. Nonetheless, and this is the surprising feature of Theorem 6, the intensity of the peer group effect can still be measured.

It is worth noting that exclusion restrictions on $A$ can create additional identification opportunities because it provides more equations with which to tie down . In large social networks where each individual is connected to a small number of neighbors, *a priori* knowledge of the location of some $A$'s zeroes quickly leads to an over-identified system.

are interviewed about their friends, among many other characteristics. Unfortunately, the data set's friendship questions are restricted in that each student is allowed to name

results on parameter identification, differing in their hypotheses about *a priori* knowledge. We then discuss contemporary econometric techniques, control functions in particular but also other instrumental variable methods, within the context of these results. We describe how control functions may be useful for parameter estimation, and also point out some potential pitfalls in the choice of instruments for interactive decision problems like Bayesian games.

### i. a group membership game

There is no one obvious network formation game to study, and so we will demonstrate the possibilities for selection in an extended example,a two-stage Bayesian game of group formation. Any strategic model of group formation must first ask, why do groups form? A distinguishing feature of social networks is the property of homophily, that similar individuals are attracted to one another. A large body of social science research (see McPherson et al. 2001) has documented that individuals in a social network are more likely to be directly connected to similar others. The urtext of sociological research on homophily, Lazarsfeld and Merton (1954), distinguishes two types of homophily, differing in their notions of similarity of individuals: *Status homophily* is the tendency for individuals to associate with those carrying similar markers of social status, such as age, ethnicity, gender, race, and income. *Value homophily* is the tendency of individuals associate with those who share common beliefs and values, regardless of their social status. The model we present below attempts to measure both of these pressuaps ( Td [(A)1(-8(c)6(y)]TJ /TT0 1(dl)6(1t)12(p(ot)2(h)10-5(3)6(v)uor)7(k(ci)1s(hei)

**a. the game**

The player set remains, as before, $V$, and type space remains the same as well, except that we will now dispense with , the characteristic observable by those in the network but not by the econometrician. An individual now has two choices in the game. In the first stage, the individual chooses a location, $a$ or $b$. In the second stage he plays the game described in Section 1 with everyone at his location, choosing $_i$ as before. At the end of the first stage, a group of people has formed at each location. We associate to each possible group $g$ the sociociomatrices $A^g$ and $C^g$. These are given exogenously, and are known *a priori* by the individuals in $V$ and by the econometrician. Since this is just an **e**xtended example, we will simplify the discussion by choosing a particular $A^g$ and $C^g$. We will assume for contextual effects that $x$ is averaged equally over all individuals in *both* groups. We will assume for peer effects that individual $i$ averages equally over all individuals other than himself in *his* own group. Peer effects are group-specific but contextual effects are not. This is a case where we would expect all parameters to be identified were there no endogeneity problem. We will also assume that the status-homophily term, with coefficient , weights according to $A^g$.

The payoff function is not yet completely specified, because it does not say what payoffs should be when $i$ is in a group of one. We will assume that in this case, $i$'s social payoff is what they would receive were they in the other group. In other words, one cannot be a group of one.

In this payoff function there are two sources of homophily. If is large relative to , affiliation will be characteristic-based. This corresponds roughly with what is meant by status-homophily. If is small relative to , affiliation will be outcome based. Individuals who desire to behave in similar ways will be more likely to group together — roughly speaking, value homophily. Value-homophily as the source of endogeneity problems. Fix a value of , and now perform the experiment of making very large. For large enough group membership is almost entirely determined by the direct effect of the publicly observable characteristics. The conditional probability distribution of group formation given $x$ converges to a point mass as diverges. Imagine the limit: For all but a measure-0 set of $x$ values, the participation conditions defining group participation hold strictly. This means that a given group is stable under small perturbations of $x$. This is enough to recover $B$, and identification proceeds as in previous sections. It is important to note that not all sources of endogeneity lead to identification issues.

The action stage of the game requires a strategy profile $f^g$ for every possible group that could form. A strategy profile for the first stage is an assignment of each individual to a location, a map $\hat{}_i\ x, _i \quad a,b$ . The assignment of individuals to locations maps each $x, _i$ to a partition of $V$ into two sets (one of which may be empty). We do not need to keep track of the locations, only the partition. Define $x,$ to be the map to partitions defined by $\hat{}$. For any partition $g,h$ the set $x, \quad ^1 g,h$ is a product set in $| |$

and construct a strategy profile $\hat{}$ which would induce it. We will call the map an *assignment*, since it allocates individuals to groups.

The *interim payoff* to $i$ for belonging to group $g$ when all $j \quad g / i$ chosen according to the strategy profile $f^g$ is

$$V_{ig}\ x,\ _i$$

$$\sup_{\ i}\ x_i \quad c_{ij}^g x_j \quad {}_i \quad {}_i \quad \frac{1}{2}\ {}_i^2 \qquad (11)$$

$$\frac{}{2}E \quad {}_i \quad {}_{j\ g/\ i} a_{ij}^g f_j^g\ x,\ {}_j \Big|^2 \Big|\ {}_i\ x,\ {}_i \quad g / i \quad \frac{}{|g|\ 1}\ x_i \quad {}_{j\ g/\ i} a_{ij}^g x_j \quad {}^2 .$$

With interim payoffs in hand, we can define a perfect Bayes equilibrium of the two-stage game.

**Definition 3.** A profile , $f^g$ $_{g\ P\ V\ /}$ is a perfect Bayes equilibrium if and only if

i. Each $f^g$ is a Bayes-Nash equilibrium of the second stage game for some conjectured $_{i\ \ i\ g}$.

ii. If $g$ occurs with positive probability and $i \quad g$, then $_i \quad E \quad {}_i \big| \quad x, \quad g$ .

iii. For each $x$ and $g \quad P\ V\ /$ containing $i$, on the event : $x, \quad g,V / g \quad , V_{i,g}\ x,\ {}_i \quad V_{i,V/g}\ {}_i\ x,\ {}_i$

The first condition says that actual action choice in groups formed, and equilibrium conjectures for groups that do not form, are the Bayes-Nash equilibrium for that group for some conjectured assignment of individuals to groups. The second condition says that beliefs about the assignment have to be correct on the equilibrium path. The third

condition is a participation constraint. It says that no individual wants to change groups given the second-stage conjectures about group choices. We shall be computing equilbria which are symmetric in that two individuals in the same choice situation, that is, the same $x$, $_i$, contextual effect, and expected peer effect, will choose the same way.

We will not prove existence here, but we will provide some characterization.[16] The characterization lemma states that if the unconditional (first-stage) expected value of the average choice of group $g$ exceeds that of group $V/g$, and individual $i$ with a given private type prefers $g$ to $V/g$, then he will prefer $g$ to $V/g$ for all higher private types. That is, assignment rules in a stable equilibrium have a threshold property.

**Lemma 4.** If $0$ and, for individual $i$ in equilibrium[17]

$$E_{j\ g}\left[a_{ij}^{g}\right]_j \qquad E_{j\ V/g}\left[a_{ij}^{V/g}\right]_j$$

and if individual $i$ with type $_i$ weakly prefers $g$ to $h$, then individual $i$ with type $_i$ $_i$ will strictly prefer $g$ to $h$. If $0$, individual $i$'s group choice is determined solely by the direct homophily effect. He will join the group with characteristics most similar to his own. If $0$ as well, then each individual is indifferent over group choice.

The consequence of this Lemma is is that for each $g$, the set of $\mathbf{R}^{|V|}$ for which $g$ forms is the product of intervals where each interval is either of the form $_i^g$, or $,\ _i^g$. The source of the selection problem is similar to that which arises in discrete choice models. Selection is determined by a threshold in the space of private types,

[16]In general it is hard to prove the existence of a perfect Bayes equilibrium for games

this threshold will change as we change individuals' observed characteristics, and so the mean of the private type of individual $i$ conditional on being in group $g$ will depend on the values of $i$'s characteristics and the characteristics of the other group members.

**b. identification**

A structure for this game is a list with elements described in Definition 4 below. We have already described the sociomatrices above. We have added the homophily parameter , and we have dispensed with . The model $M$ of this section maintains properties iii and vi of Definition 1. Unfortunately, we have, so far, found no way to identify all of the parameters of the model without assuming *a priori* knowledge of . The definition of structures and models for this section reflects all this:

**Definition 4.** A structure is a list $\langle \ , \ , \ , \ ,C,A, \ , \ \rangle$ satisfying Axioms 1 through 4. Denote by $M_{end}$ the set of structures satisfying the following additional properties:

i. $^{X}$ is finitely exchangeable.

ii. has a strictly positive density on $\mathbf{R}^{|V|}$.

iii. $_{i}$ is an independent and identically distributed sequence.

iv. For all $g$, $A_{ij}^{g}$ $\frac{1}{|g| \ 1}$ if $i$ and $j$ $i$ are both in $g$, and 0 otherwise.

v. $c_{ij}$ $\frac{1}{|V| \ 1}$.

vi. One of and is not 0.

vii. One of and is greater than 0.

An observation in this model is a triple $g, V / g, x$, where $g$ and $V / g$ are the two groups that form, $x$ is the vector of characteristics of individuals in $V$ and is the vector of their actions. An observation is an *equilibrium outcome*. We will assume that econometricians have access to all data. That is, econometricians see who is in which group, and what each individual chooses. In other words, the econometrician sees a particular equilibrium assignment of individuals to groups, and the subsequent second stage equilibria for the (no more than) two groups that formed. Thus the identification question concerns probability distributions on triples of the form $g, V / g, x$, .

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iii. If is known *a priori* and 0 , then is identified.

The difficulty with identification is recovering the matrix $B\ m$ , where all parameters except are hiding. When the network is exogenous, varying the $x$ 's a bit and seeing what happens uncovers the linear relation between $x$ and , in other words, $B\ m$ , and identification proceeds from there. When networks are endogenous, varying $x$ changes the participation constraints. The conditional mean of $_i$ given $x$ in each group move with $x$ . Perturbations in $x$

$$_{i} \quad s_{i} \quad _{i} \tag{13}$$

where

We have said nothing about how to construct the control function or whether they even exist. It is now understood that control function may not exist in certain contexts (Blundell and Matzkin (2010)). Our goal is simply to establish how one could in principle use endogenous network formation to facilitate identification of our general social

To see the import of this argument, suppose it is the case that there is additional information $z$ that affects the choice of networks in the first stage of the game but has no effect on the payoffs associated with the choices $_i$ in the second stage of the game. At first glance, one might believe that $z$ represents a set of instruments available for overcoming endogeneity of the group choices, that may be used to overcome the

information on social network structure and highlight the importance of data collection as an integral part of efforts to identify economically interesting phenomena. At the same time, our

possibility. For example, if network membership is associated with prices, then prices can help to uncover social effects, as demonstrated in recent advances in the econometrics of hedonic models (Ekeland, Heckman, and Nesheim (2004), Nesheim (2002)). One of the major themes in James Heckman's research that endogeneity is not so much a nuisance to empirical work, but rather an additional behavior that needs to be modeled. So our last suggestion is nothing more but an acknowledgement of the importance of this particular Heckman insight to future social networks research.

## Technical Appendix

### Section 2.

### Proof of Theorem 1.

$$E\ u_i \quad {}_i, \quad {}_i \left| x, \ , \ {}_i \right.$$

$$x_i \quad {}_j c_{ij} x_j \quad {}_i \quad {}_i \quad {}_i \quad \frac{1}{2} \ {}_i^2 \quad \frac{}{2} E \quad {}_j a_{ij} \ {}_j \quad {}_i \left| x, \ , \ {}_i \right.^2 .$$

Let ${}_i \quad x_i \quad {}_j c_{ij} x_j \quad {}_i \quad {}_i$ and let ${}_i^{dev}$ denote the deviation of ${}_i$ from its mean. The common knowledge assumption implies that the vector is known to all network members, so it is only the deviation from the mean that is private. Rewriting,

$$E\ u_i \quad {}_i, \quad {}_i \left| x, \ , \ {}_i^{dev} \right. \quad {}_i \quad {}_i^{dev} \quad {}_i \quad \frac{1}{2} \ {}_i^2 \quad \frac{}{2} E \quad {}_j a_{ij} \ {}_j \quad {}_i \left| x, \ , \ {}_i^{dev} \right.^2 .$$

The first-order conditions are

$${}_i \quad {}_i^{dev} \quad {}_i \quad {}_i \quad {}_j a_{ij} E \quad {}_j \left| x, \ , \ {}_i^{dev} \right. \quad 0$$

and so

$${}_i \quad \frac{1}{1} \quad {}_i \quad \frac{}{1} \quad {}_j a_{ij} E \quad {}_j \left| x, \ , \ {}_i \right. \quad \frac{1}{1} \quad {}_i^{dev}$$

which implies

$$E_i|x, \quad \frac{1}{1} \; {}_i \; \frac{}{1} \; {}_j \; a_{ij} E_j|x, \quad .$$

Thus

$$E \quad |x, \quad \frac{1}{1} \; I \; \frac{}{1} A^{\;1} \quad .$$

This means that

$$\frac{1}{1} \quad \frac{}{1} A \frac{1}{1} \; I \; \frac{}{1} A^{\;1} \quad \frac{1}{1} \; dev$$

Rearranging terms, the set of choices follows

$$\frac{1}{1} \; I \; \frac{}{1} A^{\;1} \quad \frac{1}{1} \; dev$$

$$\frac{1}{1} \; I \; \frac{}{1} A^{\;1} \quad I \quad C \; x \quad \frac{1}{1} \; dev$$

which is therefore an equilibrium.

Uniqueness, of equilibrium is proven by that the first order conditions define a contraction map on the space of strategy profiles topologized with the product $L_2$ norm.

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A fixed point of this map is a strategy profile that satisfies the first-order condition for every agent, and hence is a Bayes-Nash equilibrium. A straightforward computation shows that $T$ is a contraction mapping, and so its fixed point is unique.

**Section 4.**

**Proof of Lemma 2.**

From equation (4) and the hypothesis of the lemma it follows that

$$1 \quad B \quad AB \quad I \quad C \qquad (16)$$

for any , , $^{1}$ $B$ . Choose an $i, j$ pair satisfying the hypothesis of the lemma. The right hand side of (16) is 0, and so

$$\frac{}{1} \quad \frac{b_{ij}}{AB_{ij}}.$$

(Note that the denominator on the right cannot be 0, or else from (16), 1 which is satisfied by no model in $M$.) Thus if $m$ $m$ $B$, then . From property ii of the Definition 1 for $M$, there is an $i, j$ pair with $i$ $j$ such that $c_{ij}$ 0, and so from (16), , and the equation for any diagonal pair implies that . From these equalities it follows that .

**Proof of Theorem 2**.

Suppose that $m$ $m$ $B$. We can write

$$B \quad \frac{1}{1} \; I \quad \frac{}{1} A^{\;1} \quad I \quad C$$

$$\frac{1}{1} \; I \quad \frac{}{1} A \quad \frac{}{1} \; A^{2} \quad I \quad C$$

Since the peer-effects network is connected, some power of $A$ is strictly positive. Suppose that $m$ is such that $0$. Then $1 \quad {}^{1} I \quad / 1 \quad A^{\;1}$ is strictly positive. The assumption on $C$ of no zero columns ensures that $A^{n}C \quad 0$ if $A^{n} \quad 0$.[21] Thus $I \quad / 1 \quad A^{\;1}C$ is strictly positive. Choose an $i$ and $j$ for which $c_{ij} \quad 0$. The set of all , pairs that can make $b_{ij} \quad 0$ is a 1-dimensional line in $\mathbf{R}^{2}$. If , is not on this line, then according to Lemma 1, $m \quad m$. The set of , , triples for which $b_{ij} \quad 0$ is a closed, two-dimensional semi-algebraic set[22]; the set of models in $M^{AC}$ with parameters outside this set is generic. Thus we have generic identification of , and for any $B \quad m$ with $0$. If $0$, then generically $0$ (a consequence of the preceding argument). In this case it is straightforward to see that and . Finally, in either case, if everything else is equal, it follows that .

$$b_{ij} \quad \frac{1}{1} \quad {}_{k\ V_2^C\ V^A} \quad I \quad \frac{1}{1} A^{\ 1}_{ik} \quad c_{kj} \ .$$

Transitivity implies that the component $V_2^C$ is a clique—completely connected—and so this sum is positive. But $c_{ij}$ 0 by assumption ($i$ is not in $V_2^C$). Thus the hypothesis of Lemma 2 is satisfied.

**Proof of Lemma 3**.

Identification holds if and only if, for each matrix $B$, $^1 B$ generically produces unique parameters. So suppose $m$ $m$ $B$. Then

the last two terms implies that . Conversely, suppose that the matrices are linearly dependent, and suppose that that

$$aI \quad bC \quad cA \quad dAC \quad 0$$

for $a$, $b$, $c$ and $d$ not all 0. We will construct two (in fact, many) models $m$ and $m$ which give rise to the same $B$. For any , let $r$ $\big/$ 1 . If two models $m$ and $m$ cannot be distinguished, the following equations must be satisfied:

$$\begin{array}{rrr} & & a \\ & & ' \ b \\ c & r & r \\ d & r & r \ . \end{array}$$

Choose any $r$ $r$ in [0, 1). Substitute the first two equations into the last two to get

(since $/ 1 \quad 1$). If $0$, $m \quad m$ implies that $I \quad C \quad I \quad C$, and identification holds.

3. The case $A \quad C$ is proved in Bramoullé, Djebbari and Fortin (2009).

**Proof of Corollary 1**.

This is case 2 of Lemma 2, $AC \quad C$

$$\frac{}{1} \quad \frac{}{1}$$

Since is non-zero, the corresponding cannot equal 0, and it follows that

$$I \quad \frac{}{1} A \quad I \quad \frac{}{1} A ,$$

which immediately implies

$$\frac{}{1} A \quad \frac{}{1} A .$$

From the observation that the rows of both $A$ and $A$ sum to 1, one may conclude that , and therefore , and $A \quad A$ . Finally from these facts it follows that is the same in both models.

**Proof of Theorem 6.**

Let $B \quad M^{C}_{I \quad C^{1}}$ . It follows from equation (4) that $B$ has constant row sums, which we will call $b$. A computation show that $b$ . Rewriting (4),

$$1 \quad I \quad A \quad b \quad B^{1} \quad CB^{1} \qquad (19)$$

Consider the right-hand side as a function of . Since $A$ has 0's on the diagonal, it follows that there must be at least one value of for which all the diagonal elements of the matrix on the right are equal. Since the right-hand side is linear in , equality of the diagonal elements is true for either one or all . Choose any for which the

diagonal elements are equal. Then $1 \quad b \quad B^{1} \quad CB^{1}{}_{11}$, proving the first claim of the theorem. Next, it is easy to verify that for generic $C$, the which makes the diagonal elements identical is unique. This identifies , and then $b$ .

**Proof of Theorem 7.**

A calculation shows that that $D_{,A,C}$ (The derivative map of with respect to , $A$, and $C$) is surjective for all models in the interior of $M$, and so the implicit function theorem implies that if , , ,$A$,$C$ $B$, and , is sufficiently near to , , there are parameters , $A$ and $C$ such that , , ,$A$ ,$C$ $B$.

**Section 6**

**Proof of Lemma 4.**

The direct homophily effect, scaled by , has no effect on individuals' preferences. Let $g_1$denote the set-valued random variable whose values are the members of group 1 realized at the end of the first stage. The value of being in a group 1 conditional on $g \quad g$ is

$$V_{ig}$$

$$\frac{1}{2\ 1} \quad {}^{2}E^{\ -}{}_{i} \mid g^{\ 2} \quad 1 \quad E^{\ -2}{}_{i} \mid g \quad 2E^{\ -}{}_{i} \mid g \quad x_i \quad {}_i \quad x_i \quad {}_i{}^{2}$$

$$_{j\ g}\ x_j \quad x_i{}^{2}$$

where $\bar{\ }_i$ is the average choice of members of $g$. (This comes from the first order conditions, and substituting back.) The utility difference between $g$ and $h \quad V / g$ is

$$V_{gh} \quad \frac{}{2\ 1}\ 2\ E\ \bar{\ }_i \mid g \quad E\ \bar{\ }_i \mid h \quad {}_i$$

$$2\ E\ \bar{\ }^2{}_i \mid g \quad E\ \bar{\ }^2{}_i \mid h \qquad \text{var}\ \bar{\ }_i \mid g \quad \text{var}\ \bar{\ }_i \mid h$$

$$\sum_{j\ g} x_j \quad x_i{}^2 \qquad \sum_{j\ h} x_j \quad x_i{}^2$$

Now take expectations over $g$ to see that if $E\ \bar{\ }_i \mid g \quad E\ \bar{\ }_i \mid h \quad 0$, then $V_{gh}$ is increasing in ${}_i$, and so $g$ is preferred to $V/g$ whenever ${}_i$ is large enough.

**Proof of Theorem 8.**

We prove this by solving the game for a several different choices of $x$, and using the answers together to make inferences about parameter values. First, consider identical $x_i$'s for every individual, say $x_i \quad 1$. It is straightforward to show that there is a common cutoff $^*$ such that individuals with ${}_i \quad ^*$ go to, say, location $a$, and the remainder go to $b$. For the individual who choose location $z$,

The expectations on the left are known to the econometrician. The probability of appearing at $a$ is $\Pr_i^*$. Thus we have a third equation

$$\Pr_i^* E_{a\,i} \quad \Pr_i^* E_{b\,i} \quad E_i .$$

If the econometrician knows

$$E_{a\ \ 1}\,|\,x_1\quad k\quad k\quad z_a^k$$

where $z_k$ can be computed from what is observable and the *a priori* knowledge of $E\ \ _i$. Now repeat the construction with individual 1 at location $b$ to get

$$E_{b\ \ 1}\,|\,x_1\quad k\quad k\quad z_b^k$$

Since the probability of individual 1 appearing at each location is known, expect over the location to derive

$$E\ \ _1\quad k\quad \Pr\ \ _1\quad {}^{*}_{1}\ z_a^k\quad \Pr\ \ _1\quad {}^{*}_{1}\ z_b^k,$$

and so can be computed. Knowing gives , which proves the claim for $M_{end}$ with known $E\ \ _1$.

If the conditional expectations are known, take any group $g$ with more than 1 person that forms with positive probability given $x$. Then $E\quad |\,x,g\quad E\quad |\,x,g$ is linear in $x$. The probabilities of entry into $g$ are continuous in $x$, so this difference in fact is well-defined on an open set around $x$. From this infer $g$ to get $m$. The result now follows from the proof of theorem 2.

The parameter has no role in the second stage game. It determines only the probabilities of group formation. Consequently must be identified off the participation constraint, that the ex-ante expected value of going to location $a$ is at least that of going to $b$ for those who chose to go to $a$, and so forth. The threshold in any game is determined by the equality of the expectation of the interim payoffs over which groups

knowledge of , because the probabilities of a given individual appearing at either location in equilibrium are known, and the distribution function of $_i$ is strictly increasing. Then the equilibrium condition defining $^*$, that when $_i$ $^*$ the individual is indifferent (ex-ante) between *a* and *b*, identifies . The prior belief is needed not just to determine $^*$, but also because the participation constraints involve differences in variances of the $_i$ conditional on location, and there are not enough equations to pin these down even given *a priori* knowledge of the unconditional variance of the $_i$.

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